McGILL UNIVERSITY

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATH 263

ORDINARY DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

Examiner: Professor J.J Xu
Associate Examiner: Dr. H. Huang

Date: Wednesday December 9, 2009
Time: 9:00 AM- 12:00 PM

INSTRUCTIONS

1. Please answer all questions in the exam booklets provided.
2. This is a closed book examination. No books, crib sheets or lecture notes permitted.
3. Faculty Standard calculators are permitted.
4. Dictionaries are not permitted.

This exam comprises the cover page, 2 pages of 8 questions and a page of the Table of Laplace Transforms.

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(Laplace Table is included)

1. **(10 Points)** Find the general solution of

$$\frac{dy}{dx} = -\frac{y^2 + 2xy}{x^2 + 2xy - 1}$$

2. **(10 Points)** Solve the following initial value problem

$$y'' + 2y' - 3y = 0, \quad y(0) = 4, y'(0) = 0$$

3. **(15 Points)** Find the general solution of

$$x^2y'' - 2xy' + 2y = x^2, \quad (x > 0),$$

by using the **method of variation of parameters.**

4. **(10 Points)** Consider the differential equation

$$t^2y'' + 2ty' - 2y = 0$$

(a) Verify that $y_1(t) = \frac{1}{t^2}$ is a solution of the differential equation.

(b) Find a second linear independent solution of the differential equation by the **reduction of order method.**

5. **(15 Points)** Solve the following initial value problem by using the **method of Laplace transform:**

$$y''' - y'' = e^t, \quad y(0) = 0, \; y'(0) = 0, \; y''(0) = 1$$

6. **(15 Points)** Given the following matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) derive the rank of matrix;

(b) find all the eigenvalues $\{\lambda_i\}$ and determine the multiplicity of each eigenvalue;

(c) calculate the eigenvectors corresponding to each eigenvalue, and determine the dimension of each eigen-space;

(d) state whether the matrix is defective or non-defective;

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(e) give the basis of each eigen-space.

7. **(15 Points)** For the matrix A given **in the problem** (6), find the general solution of the homogeneous EQ.

$$\mathbf{x}' = A\mathbf{x}.$$

8. **(10 Points)** With the matrix A given **in the problem (6)**, find the solution for the IVP of non-homogeneous EQ:

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b},$$

where $\mathbf{b} = (t^2, t, 1)^T$ and the initial **condition** $\mathbf{x}(0) = \mathbf{x}_0 = (-2, 1, 1)^T$.

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\dfrac{1}{s}, \quad s > 0$
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$
3. t^n; $\quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \; p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
9. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}$, $n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t) f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct} f(t)$	$F(s-c)$
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$